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Some Polynomials Associated with the Generalized Hermite Polynomials

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I.

Like Carlitz [1–3], Singh [5], K. N. Srivastava [7] and others, we introduce the associated generalized Hermite polynomials by means of

$$\sum_{k=0}^{n} \frac{1}{(n-k)!} A_k^{(\alpha)}(x, r, p)\, G_{n-k}^{(\alpha-k)}(x, r, p) = 0, n \geqslant 1, \tag{1.1}$$

and

$$A_0^{(\alpha)}(x, r, p) = 1, \tag{1.2}$$

where

$$G_n^{(\alpha)}(x, r, p) = (-1)^n x^{-\alpha} \exp(px^r)\, D^n\{x^\alpha \exp(-px^r)\} \tag{1.3}$$

defines the generalized Hermite polynomials of Gould and Hopper [4].[1]

It follows, as a consequence of our definition that

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \frac{1}{(n-k)!} A_k^{(\alpha)}(x, r, p)\, G_{n-k}^{(\alpha-k)}(x, r, p) \\ &= \sum_{k=0}^{\infty} t^k A_k^{(\alpha)}(x, r, p) \sum_{n=0}^{\infty} \frac{t^n}{n!} G_n^{(\alpha-k)}(x, r, p). \end{aligned}$$

Therefore, in view of the fomula [4],

$$x^{-\alpha}(x-t)^\alpha \exp[p\{x^r - (x-t)^r\}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} G_n^{(\alpha)}(x, r, p), \tag{1.4}$$

[1] Although for reasons of convenience the notation used here is slightly in variance with that of Gould and Hopper, the polynomials are essentially the same.

we readily obtain

$$\left(1+\frac{z}{x}\right)^{\alpha} \exp\left[p\left\{-x^r+\frac{x^{2r}}{(x+z)^r}\right\}\right] = \sum_{k=0}^{\infty} z^k A_k^{(\alpha)}(x, r, p), \quad (1.5)$$

where $t(x + z) = xz$.

On rewriting Eq. (1.5) in the form

$$\sum_{k=0}^{\infty} A_k^{(\alpha)}(x, r, p)\, z^k = \exp(-px^r) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n \frac{(mr-\alpha)_n}{n!\, m!} p^n z^n x^{mr-n}$$

and equating the coefficients of z^n, we derive the explicit formula

$$A_n^{(\alpha)}(x, r, p) = \frac{\exp(-px^r)}{n!} \sum_{m=0}^{\infty} (-1)^n \frac{(mr-\alpha)_n}{m!} p^m x^{mr-n}. \quad (1.6)$$

Since

$$(-1)^n (mr-\alpha)_n = (\alpha - n + 1 - mr)_n,$$

the above equation simplifies to

$$A_n^{(\alpha)}(x, r, p) = \frac{\exp(-px^r)}{n!} \sum_{m=0}^{\infty} \frac{(\alpha-n+1-mr)_n}{m!} p^m x^{mr-n}, \quad (1.7)$$

and this can also be put in the elegant form

$$A_n^{(\alpha)}(x, r, p) = \frac{(\alpha-n+1)_n \exp(-px^r)}{n!} \sum_{m=0}^{\infty} \frac{(n-\alpha)_{mr}}{(-\alpha)_{mr}} \frac{p^m x^{mr-n}}{m!}. \quad (1.8)$$

Next, from (1.4), we notice that

$$G_n^{(\alpha)}(x, r, p) = \exp(px^r) \sum_{m=0}^{\infty} \frac{(-\alpha-mr)_n}{m!} (-p)^m x^{mr-n}, \quad (1.9)$$

so that on suitably adjusting the parameters in (1.7) we get

$$A_n^{(\alpha)}(x, r, p) = (1/n!)\, G_n^{(n-\alpha-1)}(x, r, -p), \quad (1.10)$$

and hence, in a straightforward manner, from (1.2) we deduce the Rodrigues formula,

$$A_n^{(\alpha)}(x, r, p) = ((-1)^n/n!) \exp(-px^r)\, x^{\alpha-n+1} D^n\{x^{n-\alpha-1} \exp(px^r)\}, \quad (1.11)$$

for the associated generalized Hermite polynomials.

II.

Consider the product

$$\left(\sum_{n=0}^{\infty} z^n A_n^{(\alpha)}(x, r, p)\right)\left(\sum_{k=0}^{\infty} z^k A_k^{(\beta)}(y, r, p)\right)$$
$$= (1 + z/x)^{\alpha} (1 + z/y)^{\beta} \exp\{-x^r - y^r + (x^{2r}/(x + z)^r) + (y^{2r}/(y + z)^r)\},$$

so that when $x = y$,

$$\sum_{m=0}^{\infty} z^m A_m^{(\alpha+\beta)}(x, r, 2p)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} z^{n+k} A_n^{(\alpha)}(x, r, p)\, A_k^{(\beta)}(x, r, p), \tag{2.1}$$

and, therefore,

$$A_n^{(\alpha+\beta)}(x, r, 2p) = \sum_{k=0}^{n} A_k^{(\beta)}(x, r, p)\, A_{n-k}^{(\alpha)}(x, r, p), \tag{2.2}$$

which admits a generalization in the form

$$A_n^{(\alpha+\beta)}(x, r, p + q) = \sum_{k=0}^{n} A_k^{(\beta)}(x, r, p)\, A_{n-k}^{(\alpha)}(x, r, q), \tag{2.3}$$

thereby providing an inverse for (3.10) of [4].

Next, since

$$x^{-\alpha}(x - t)^{\alpha} \exp[p\{-x^r - (x - t)^r\}]$$
$$= \sum_{k=0}^{\infty} \left(\frac{xt}{x - t}\right)^k A_k^{(\beta)}(x, r, p)\left(1 - \frac{t}{x}\right)^{\alpha+\beta} \exp[2p\{x^r - (x - t)^r\}],$$

by an appeal to (1.4), it follows that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G_n^{(\alpha)}(x, r, p) = \sum_{k=0}^{\infty} t^k A_k^{(\beta)}(x, r, p) \sum_{n=0}^{\infty} \frac{t^n}{n!} G_n^{(\alpha+\beta-k)}(x, r, 2p), \tag{2.4}$$

and finally,

$$G_n^{(\alpha)}(x, r, p) = \sum_{k=0}^{n} \frac{n!}{(n - k)!} A_k^{(\beta)}(x, r, p)\, G_{n-k}^{(\alpha+\beta-k)}(x, r, 2p), \tag{2.5}$$

or equivalently

$$G_n^{(\alpha)}(x, r, p) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(k-\beta-1)}(x, r, -p)\, G_{n-k}^{(\alpha+\beta-k)}(x, r, 2p). \tag{2.6}$$

It is not difficult to verify that Eq. (2.6), on generalization, yields the formula

$$G_n^{(\alpha)}(x, r, p) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(k-\beta-1)}(x, r, -q)\, G_{n-k}^{(\alpha+\beta-k)}(x, r, p+q), \tag{2.7}$$

for the generalized Hermite polynomials. Now, consider the sum

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n+k}{n} A_{n+k}^{(\alpha)}(x, r, p)\, z^n v^k &= \sum_{n=0}^{\infty} (z+v)^n A_n^{(\alpha)}(x, r, p) \\
&= x^{-\alpha}(x+z+v)^{\alpha} \exp\left[p\left\{-x^r + \frac{x^{2r}}{(x+z+v)^r}\right\}\right] \\
&= x^{-\alpha}(x+z)^{\alpha} \exp\left[p\left\{-x^r + \frac{x^{2r}}{(x+z)^r}\right\}\right] \\
&\quad \times \left[1 + \frac{v}{x+z}\right]^{\alpha} \exp\left[p\,\frac{x^{2r}}{(x+z)^r}\left\{\frac{(x+z)^r}{(x+z+v)^r} - 1\right\}\right] \\
&= x^{-\alpha}(x+z)^{\alpha} \exp\left[p\left\{-x^r + \frac{x^{2r}}{(x+z)^r}\right\}\right] \\
&\quad \times \sum_{k=0}^{n} \left\{\frac{vx^2}{(x+z)^2}\right\}^k A_k^{(\alpha)}\left\{\frac{x^2}{x+z}, r, p\right\}.
\end{aligned}$$

Thus on equating the coefficients of v^k, we shall obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \binom{n+k}{n} A_{n+k}^{(\alpha)}(x, r, p)\, z^n \\
&\quad = (1+z/x)^{\alpha-2k} \exp[p\{-x^r + (x^{2r}/(x+z)^r)\}]\, A_k^{(\alpha)}\{(x^2/(x+z)), r, p\}
\end{aligned} \tag{2.8}$$

or, alternatively, the identity

$$\begin{aligned}
&\sum_{n=0}^{\infty} \binom{n+k}{n} A_{n+k}^{(\alpha)}(x, r, p)\, z^n \\
&\quad = \sum_{n=0}^{\infty} z^n A_n^{(\alpha-2k)}(x, r, p)\, A_n^{(\alpha)}\left\{\frac{x^2}{x+z}, r, p\right\}.
\end{aligned} \tag{2.9}$$

III.

As usual, let δ stand for the differential operator $x(d/dx)$, satisfying the well known properties:

$$F(\delta)\{x^n\} = F(n)\, x^n, \tag{3.1}$$

$$F(\delta)[\exp\{g(x)\}f(x)] = \exp\{g(x)\}\, F\{\delta + xg'(x)\}\, f(x), \tag{3.2}$$

and

$$x^{n\alpha}\, \delta(\delta + 1) \cdots (\delta + n - 1)\, f(x) = [x^\alpha\, \delta]^n f(x). \tag{3.3}$$

Employing the formula (1.8), we have

$$\begin{aligned}
&(\delta + prx^r + n)(\delta + prx^r + n - \alpha - r)_r\, A_n^{(\alpha)}(x, r, p) \\
&\quad = \frac{r(\alpha - n + 1)_n \exp(-px^r)}{n!} \sum_{m=1}^{\infty} \frac{m(mr - \alpha - r)_r}{m!} \frac{(n - \alpha)_{mr}}{(-\alpha)_{mr}}\, p^m x^{mr-n} \\
&\quad = \frac{prx^r(\alpha - n + 1)_n \exp(-px^r)}{n!} \sum_{m=0}^{\infty} \frac{(n - \alpha)_{\overline{m+1}^r}}{m!(-\alpha)_{mr}}\, p^m x^{mr-n} \\
&\quad = prx^r(\delta + prx^r + 2n - \alpha)_r\, A_n^{(\alpha)}(x, r, p).
\end{aligned}$$

Thus, we have the differential equation,

$$\begin{aligned}
&[(\delta + prx^r + n)(\delta + prx^r + n - \alpha - r)_r \\
&\quad - prx^r(\delta + prx^r + 2n - \alpha)_r]\, y = 0,
\end{aligned} \tag{3.4}$$

satisfied by $A_n^{(\alpha)}(x, r, p)$.

It will not be out of place to remark that on replacing α by $n - \alpha - 1$, p by $-p$ and multiplying throughout by $n!$, the formula (3.4) would reduce to our earlier result ((3.4) in [6]), for the generalized Hermite polynomials.

IV.

In view of (3.1)–(3.3) and since the operators commute, it can be deduced fairly easily that

$$\begin{aligned}
&D^n[x^{n-\alpha-1} \exp(px^r)\, y] \\
&\quad = x^{-\alpha-1} \exp(px^r) \prod_{j=1}^{n} (\delta + prx^r - \alpha - 1 + j)\, y,
\end{aligned} \tag{4.1}$$

where y is a sufficiently differentiable function of x.

On the other hand, using the Leibnitz theorem and the formula (1.11), we also have

$$D^n[x^{n-\alpha-1}\exp(px^r)\,y] = (-1)^n\,n!\,x^{n-\alpha-1}\exp(px^r)\sum_{k=0}^{n}\frac{(-1)^k}{k!}A_{n-k}^{(\alpha-k)}(x,r,p)\,D^k y. \quad (4.2)$$

Therefore, comparison of (4.1) and (4.2) leads to

$$\prod_{j=1}^{n}(\delta + prx^r - \alpha - 1 + j)\,y = n!(-x)^n\sum_{k=0}^{n}\frac{(-1)^k}{k!}A_{n-k}^{(\alpha-k)}(x,r,p)\,D^k y, \quad (4.3)$$

so that when $y = 1$, we have

$$\prod_{j=1}^{n}(\delta + prx^r - \alpha - 1 + j) = (-x)^n\,n!\,A_n^{(\alpha)}(x,r,p). \quad (4.4)$$

Again, on rewriting (4.1) in the form

$$\prod_{j=1}^{n}(\delta + prx^r - \alpha - 1 + j)\,y = x^{-n+\alpha+1}\exp(-px^r)[x(xD+1)]^n\{x^{-\alpha-1}\exp(px^r)\,y\}, \quad (4.5)$$

from (4.3), we readily obtain the formula

$$[x(xD+1)]^n\{x^{-\alpha-1}\exp(px^r)\,y\} = (-1)^n\,n!\cdot x^{-\alpha-1}\exp(px^r)\sum_{k=0}^{n}\frac{(-1)^k}{k!}A_{n-k}^{(\alpha-k)}(x,r,p)\,D^k y. \quad (4.6)$$

If now we express

$$x^{\alpha-n+1}\exp(-px^r)\,D^n[x^{n-\alpha-1}\exp(px^r)\,y],$$

in the form

$$x^{\alpha-2n+1}\exp(-px^r)\,\delta(\delta-1)\cdots(\delta-n+1)[x^{n-k}x^{k-\alpha-1}\exp(px^r)\,y],$$

and make use of (3.1)–(3.3) and also of (4.2), we are led to the operational formula

$$[x(xD-k+1)]^n[x^{k-\alpha-1}\exp(px^r)\,y] = (-1)^n\,n!\exp(px^r)\,x^{2n-\alpha+k-1}\sum_{s=0}^{n}\frac{(-1)^s}{s!}A_{n-s}^{(\alpha+s)}(x,r,p)\,D^s y, \quad (4.7)$$

which corresponds to (4.6) when $k = 0$.

Setting $k = 2$, $y = 1$ and $\Phi \equiv x(xD - 1)$, the above formula assumes the form

$$\Phi^n[x^{-\alpha+1}\exp(px^r)] = (-1)^n\, n!\, x^{2n-\alpha+1}\exp(px^r)\, A_n^{(\alpha)}(x, r, p), \tag{4.8}$$

so that

$$\Phi^m[x^{2n-\alpha+1}\exp(px^r)\, A_n^{(\alpha)}(x, r, p)] = [(-1)^m\,(m+n)!/n!]\, x^{2m+2n-\alpha+1}\exp(px^r)\, A_{m+n}^{(\alpha)}(x, r, p). \tag{4.9}$$

However, if we replace n by m, α by $\alpha - 2n$ and put $y = A_n^{(\alpha)}(x, r, p)$, (4.7) would yield

$$\Phi^m[x^{2n-\alpha+1}\exp(px^r)\, A_n^{(\alpha)}(x, r, p)] = (-1)^m\, m!\,\exp(px^r)\, x^{2m+2n-\alpha+1} \cdot \sum_{s=0}^{m} \frac{(-1)^s}{s!}\, A_{m-s}^{(\alpha-2n-s)}(x, r, p)\, D^s A_n^{(\alpha)}(x, r, p). \tag{4.10}$$

Evidently, from (4.9) and (4.10), we would get

$$\binom{m+n}{m} A_{m+n}^{(\alpha)}(x, r, p) = \sum_{s=0}^{m} \frac{(-1)^s}{s!}\, A_{m-s}^{(\alpha-2n-s)}(x, r, p) \cdot D^s A_n^{(\alpha)}(x, r, p), \tag{4.11}$$

which leads to

$$\sum_{m=0}^{\infty} \binom{m+n}{m} A_{m+n}^{(\alpha)}(x, r, p)\, t^m = \left(1 + \frac{t}{x}\right)^{\alpha-2n} \exp\left[p\left\{-x^r + \frac{x^{2r}}{(x+t)^r}\right\}\right] \sum_{s=0}^{\infty} \left(\frac{-xt}{x+t}\right)^s \frac{1}{s!}\, D^s A_n^{(\alpha)}(x, r, p), \tag{4.12}$$

and on a little simplification, we obtain the generating relation (see also (2.8))

$$\sum_{m=0}^{\infty} \binom{m+n}{m} A_{m+n}^{(\alpha)}(x, r, p)\, t^m = \left(1 + \frac{t}{x}\right)^{\alpha-2n} \exp\left[p\left\{-x^r + \frac{x^{2r}}{(x+t)^r}\right\}\right] A_n^{(\alpha)}\left\{\frac{x^2}{x+t}, r, p\right\}, \tag{4.13}$$

or finally in terms of generalized Hermite polynomials, we have

$$\sum_{m=0}^{\infty} \frac{t^m}{m!}\, G_{m+n}^{(m-\beta)}(x, r, p) = \left(1 + \frac{t}{x}\right)^{\beta-n-1} \exp[p\{-x^r + (x^{2r}/(x+t)^r)\}]\, G_n^{(-\beta)}\{(xt/(x+t)), r, p\}. \tag{4.14}$$

V.

The differential and pure recurrence relations follow in a rather straightforward manner from (1.5) in the form

$$\{xD + prx^r - (\alpha - n + 1)\}\, A_n^{(\alpha)}(x, r, p) = -(n + 1)\, xA_{n+1}^{(\alpha+1)}(x, r, p), \tag{5.1}$$

and

$$\alpha A_{n-1}^{(\alpha-1)}(x, r, p) - prx^r A_{n-1}^{(\alpha-r-1)}(x, r, p) = nxA_n^{(\alpha)}(x, r, p), \tag{5.2}$$

respectively. If, however, we make use of the difference operator

$$\Delta f(\alpha) = f(\alpha + 1) - f(\alpha),$$

the formula (1.5) would yield

$$A_n^{(\alpha)}(x, r, p) = xA_{n+1}^{(\alpha+1)}(x, r, p) - xA_{n+1}^{(\alpha)}(x, r, p). \tag{5.3}$$

This implies that

$$\Delta A_n^{(\alpha)}(x, r, p) = (1/x)\, A_{n-1}^{(\alpha)}(x, r, p),$$

and, consequently,

$$\Delta^k A_n^{(\alpha)}(x, r, p) = (1/x^k)\, A_{n-k}^{(\alpha)}(x, r, p). \tag{5.4}$$

Therefore, setting $E \equiv \Delta + 1$, so that $E\{f(\alpha)\} = f(\alpha + 1)$, (1.11) would lead to

$$E^n A_n^{(\alpha)}(x, r, p) = A_n^{(\alpha+n)}(x, r, p),$$

and this in view of (5.4) simplifies to the neat formula

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{x^k}\, A_{n-k}^{(\alpha)}(x, r, p) = A_n^{(\alpha+n)}(x, r, p). \tag{5.5}$$

While concluding, we remark that numerous recurrence relations can be obtained by combining (5.1)–(5.3) and (5.5).

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